The Free Rider Problem: a Dynamic Analysis

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I. Introduction

- Most free riders problems have an important dynamic component.
- Still, there is only a limited understanding of dynamic free rider problems:
  - Research has been focused on economics with reversibility, under special assumptions;
  - There is no analysis of the classic free rider problem with irreversibility.

  - In this paper we provide a comparative analysis of Markov equilibria, with and without reversibility.

- At the core of the paper there is a new approach to characterize the Markov equilibria of a stochastic game:
  - We characterize weakly concave equilibria;
  - We show it is without loss of generality.

- With reversibity, a continuum of equilibria: the lowest decreasing in *n*; the highest increasing in *n*. The highest steady state → efficiency as $\delta \to 1$.

- With irreversability, the set of steady states converges to the highest steady state with reversibility as *d*->*0*.

- We may have monotonic or spiraling convergence; or persistent cycles.

I.3 Plan for today

I. The model

- The economy
- The planner's problem

II. Equilibria in a reversible economy

III. Equilibria in a irreversible economy

IV. Non-monotonic strategies and cycles

V. Conclusion

I. The model

II. 1 **The economy**

- Consider an economy with n agents. There are two goods: private good x and a public good g.

- We assume that U^j can be written as:

$$U^j(z^j) = \sum_{t=1}^{\infty} \delta^{t-1} \left[x_t^j + u(g_t) \right],$$

- The rate of transformation between x and g is 1.

- Private consumption good is nondurable, the public good is durable:

$$g_t = (1-d)g_{t-1} + I_t.$$

- In a *Reversible Economy* (RIE):

$$x^j \geq 0 \quad \forall j$$

$$y \geq 0$$

$$\sum_{j=1}^{n} x^j + \left[y - (1-d)g \right] \leq W$$

where $g=g_{t-1}$ and $y=g_t$ and W is the per period endowment.

- In a *Irreversible Economy* (IIE), the second constraints is substituted with: $$y \geq (1-d)\,\mathrm{g}$$

- In a RIE the public investment can be scaled back. In a IIE investment can not be undone.

- In period t, each agent i is endowed with W/n units of x.

- In each period i independently chooses how to allocate its endowment between g and x.

- In a RIE, $x^i \leq W/n+(1-d)g/n$. In a IIE, $x^i \leq W/n$.

- The economy-wide investment is the sum of the investments.

- The level of the state variable g, therefore, creates a dynamic linkage across policy making periods.

- We focus on symmetric Markov equilibria with continuous strategies: $x(g)$, $y(g)$, with associated value function $v(g)$.

II.2 **The planner's solution**

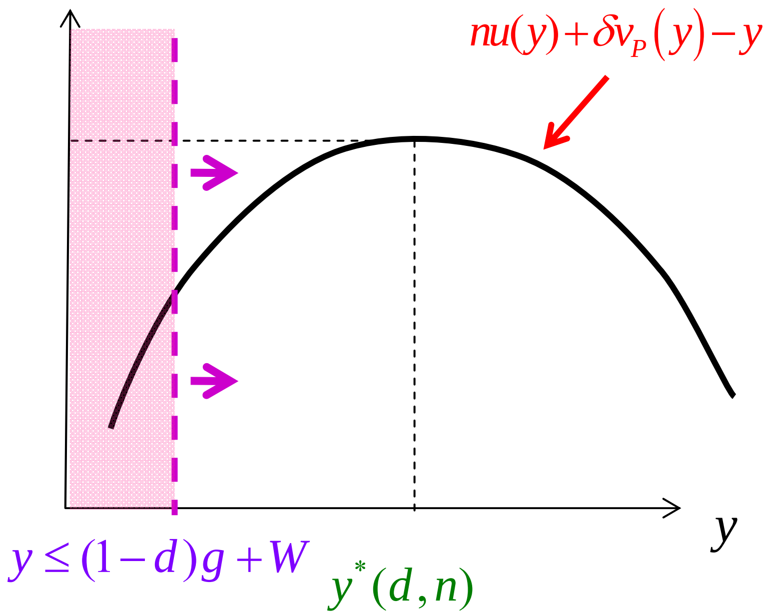
- The planner's problem has a recursive representation as:

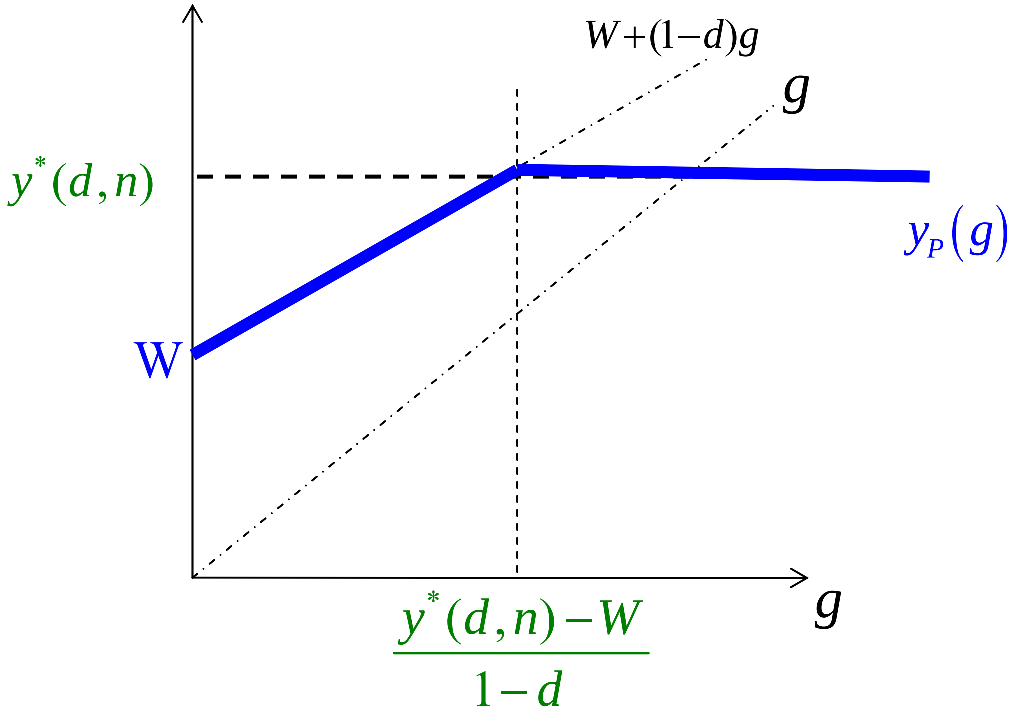
$$v_P(g) = \max_{y,x} \left\{ \begin{array}{c} \sum_{j=1}^{n} x^j + nu(y) + \delta v_P(y) \\ s.t \;\; \sum_{j=1}^{n} x^j + y - (1-d)g \le W, \\ x^i \ge 0 \;\forall i, \; y \ge 0 \end{array} \right\} \quad \textbf{(*)}$$

- By standard methods, we can show that a continuous, concave and differentiable v_P (g) that satisfies **(*)** exists and is unique.

- The optimal policies have an intuitive characterization.

- We start from the case of RIE.





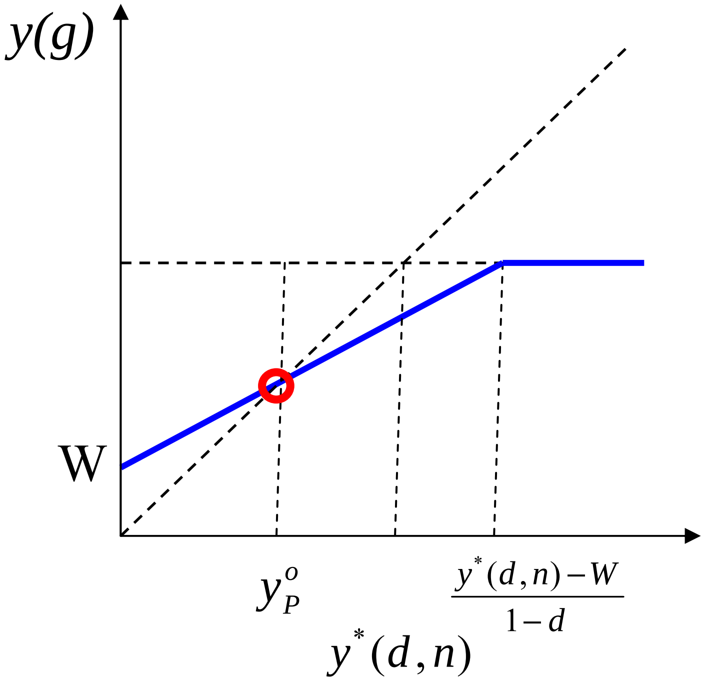
So the equilibrium investment function is:

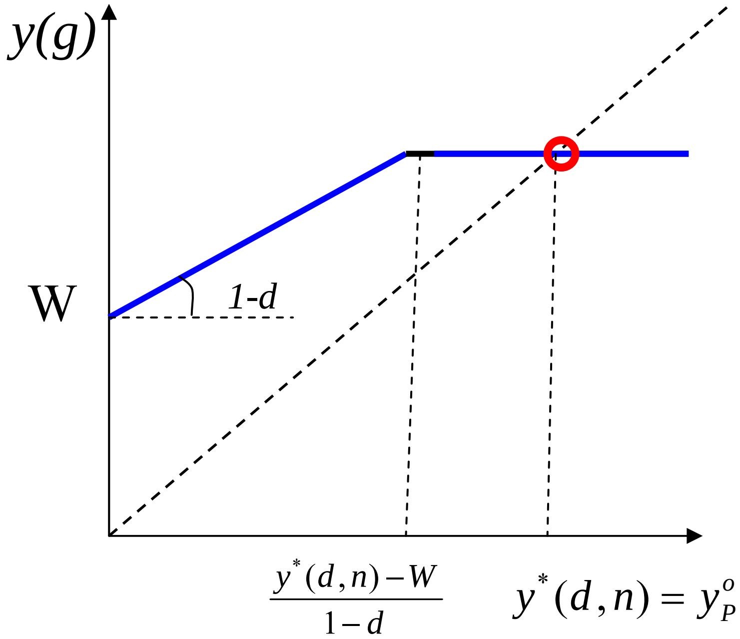
$$y_P(g) = \min\left\{W + (1-d)g,\, y_P^*(d,n)\right\}.$$

where (it can be shown):

$$y_P^*(d,n) = [u']^{-1}\left(\frac{1-\delta(1-d)}{n}\right)$$

- We can have two cases:



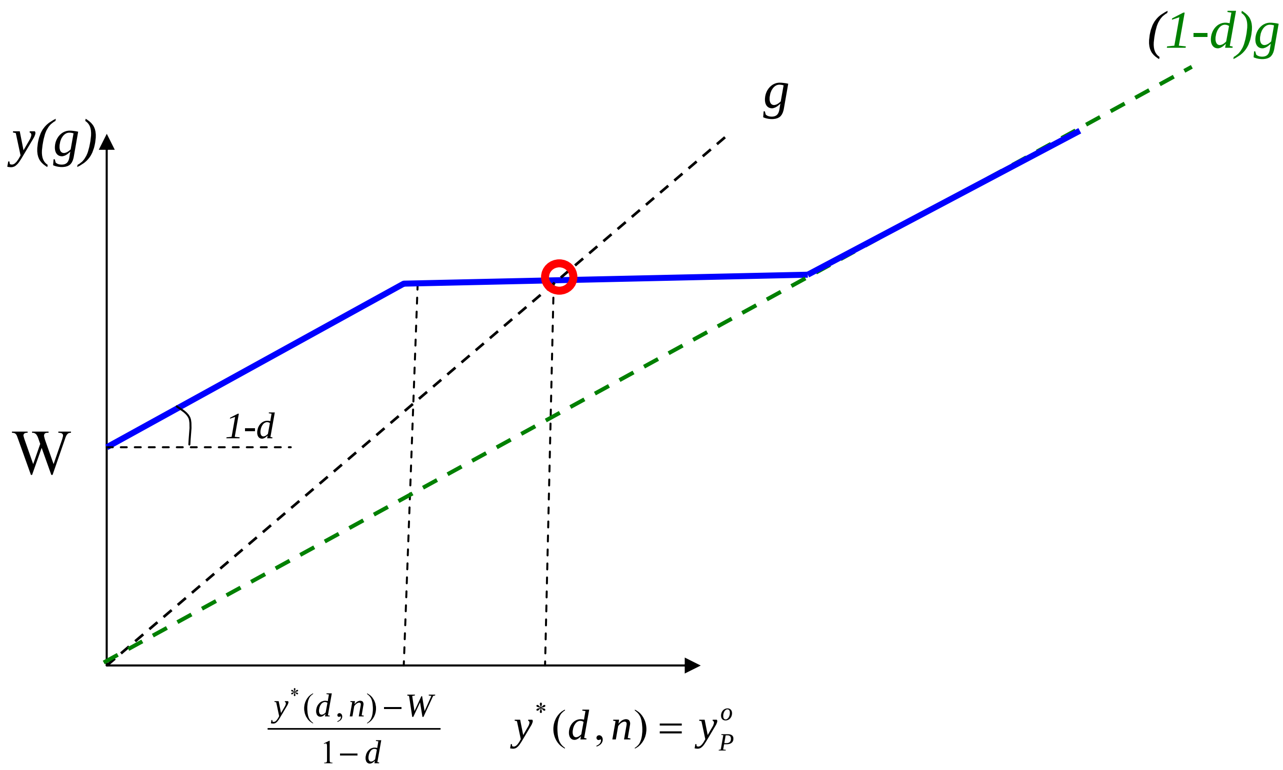


Case 1: $W/d \le [u']^{-1}(1-n\delta(1-d))$

Case 2: $W/d > [u']^{-1}(1-n\delta(1-d))$

- We focus on the second case: *regular economies*.

- The case of IIE is almost the same.



- The irreversibility constraint is irrelevant because it affects the economy only out of equilibrium.

II. Equilibria in a RIE

- The optimization problem for agent *j* in state *g* is:

$$\max_{y,x}\left\{\begin{array}{c} x+u(y)+\delta v_R(y) \\ s.t\ \ x+y-(1-d)g = W-(n-1)x_R(g) \\ W-(n-1)x_R(g)+(1-d)g-y \geq 0 \\ nx \leq (1-d)g+W \end{array}\right\}$$

Agent *j* can not choose *y* directly. Given the other agents' investments, *j*'s ultimately determines *y*.

- In a symmetric equilibrium, all agents consume the same fraction of resources:

$$x_R(g)=\frac{1}{n}\left[W+(1-d)g-y_R(g)\right].$$

- Agent's j problem is then equivalent to:

$$\max_{y} \left\{ \begin{array}{c} u(y) - y + \delta v_R(y) \\ y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y_R(g),\ y \geq \frac{n-1}{n} y_R(g) \end{array} \right\} \qquad (*)$$

- Given the agent's choice $y_R(g)$, the expected value must be:

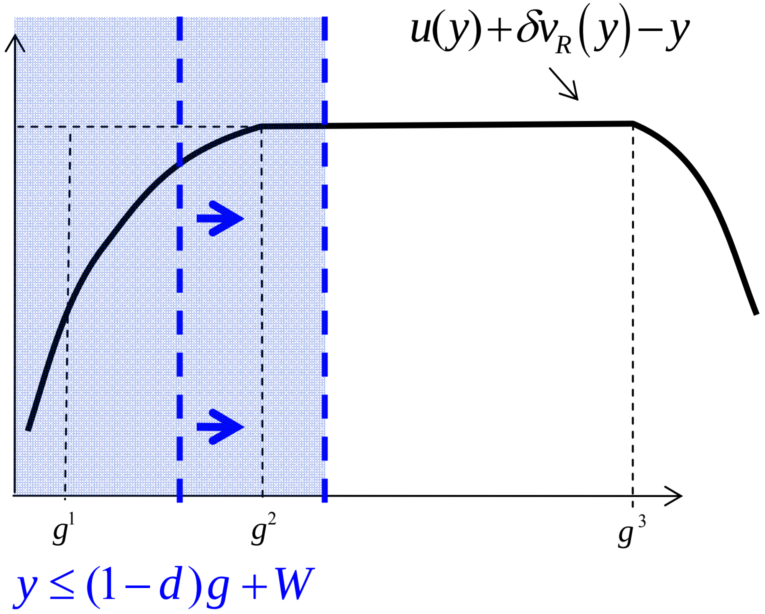
$$v_R(g) = \frac{W + (1-d)g - y_R(g)}{n} + u\big(y_R(g)\big) + \delta v_R(y_R(g)) \qquad (**)$$

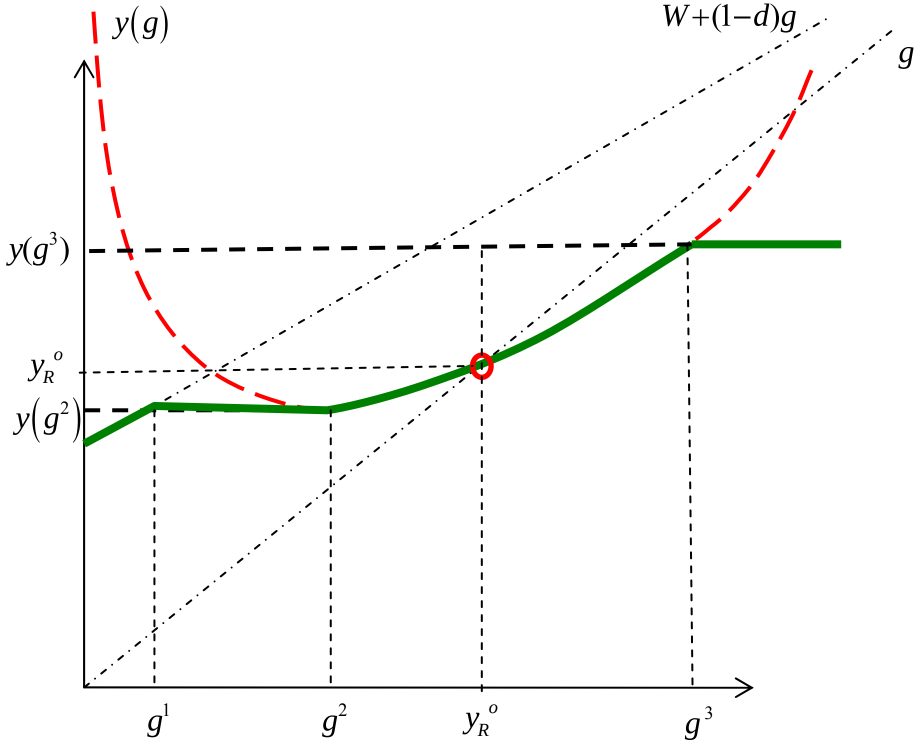
Definition. *An equilibrium in a RIE is a pair of functions $y_R(g)$ and a $v_R(g)$ such that for all g, $y_R(g)$ solves (*) given $v_R(g)$; and for all g, $v_R(g)$ solves (**) given $y_R(g)$.*

- Contrary to the planner's case, we know little a priori on *v(g)* and *y(g)*.

- Indeed, now, there is a loss of generality in focusing only on strictly concave objective functions.

- In the following:
  - I will first illustrate a class of equilibria with weakly concave functions.
  - I will then show that there is no loss of generality in using this class.





So now the investment function is:

$$y_R(g) = \begin{cases} \max\left\{W + (1-d)g, y\left(g^2\right)\right\} & g < g^2 \\ y\left(g\right) & g \in \left[g^2, g^3\right] \\ y\left(g^3\right) & g > g^3 \end{cases}$$

When is this reaction function an equilibrium?

- First, agents must be indifferent between investing and consuming for all states in $[g^2,g^3]$:

$$u'(g) + \delta v'(g) - 1 = 0 \quad \forall g \in \left[g^2, g^3\right]$$

- Since:

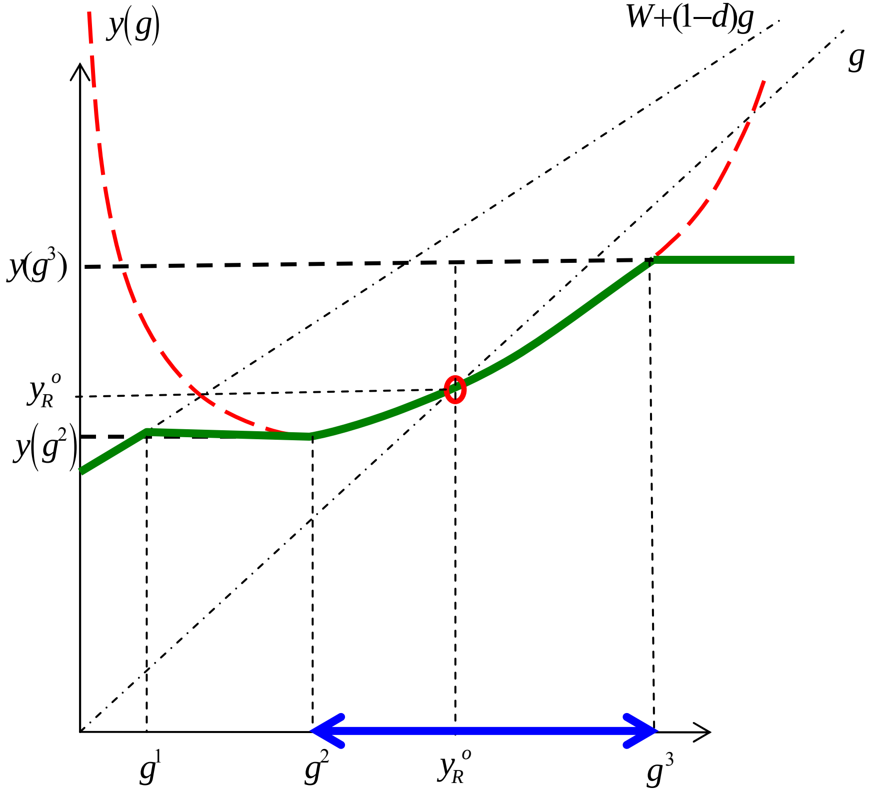
$$v(g) = \frac{W + (1-d)g - y(g)}{n} + u(y(g)) + \delta v(y(g))$$

- We have:

$$v'(g) = \frac{1-d-y'(g)}{n} + u'(y(g))y'(g) + \delta v'(y(g))y'(g)$$

- That leads to the necessary condition:

$$\frac{1-u'(g)}{\delta} = \frac{1-d-y'(g)}{n} + u'(y(g))y'(g) + \delta v'(y(g))y'(g)$$



Note:

•For g in $[g_2, g_3]$, $y(g)$ is in $[g_2, g_3]$.

$$u'(y(g)) + \delta v'(y(g)) - 1 = 0$$

- Substituting, we get:

$$\frac{1-u'(g)}{\delta} = \frac{1-d-y'(g)}{n} + u'(y(g))\,y'(g) + \left[1-u'(y(g))\right] y'(g)$$

- This uniquely defines a function $y(g)$, up to a constant.

- For our example, the terminal condition

$$y(y^0_R)=y^0_R$$

uniquely determines the investment function in $[g_2,g_3]$;

- Questions:
  - What steady states can we achieve?
  - Do the equilibria constructed in this way span the set of well behaved equilibria?

- For simplicity, we focus here on monotonic equilibria.

- An equilibrium is *monotonic* if $y_R(g)$ is monotonic non decreasing in g.

Proposition. *An investment level* $y^o{}_R$ *is stable steady state of a monotonic equilibrium if and only if:*

$$[u']^{-1}\left(1-\delta\frac{(1-d)}{n}\right)\leq y_R^o\leq[u']^{-1}\left(1-\delta\left(1-\frac{d}{n}\right)\right)$$

Each $y^o{}_R$ *is supported by a concave equilibrium with investment function as described as above.*

Properties:

- All equilibria are inefficient.

- The autarky steady state is in the interior.

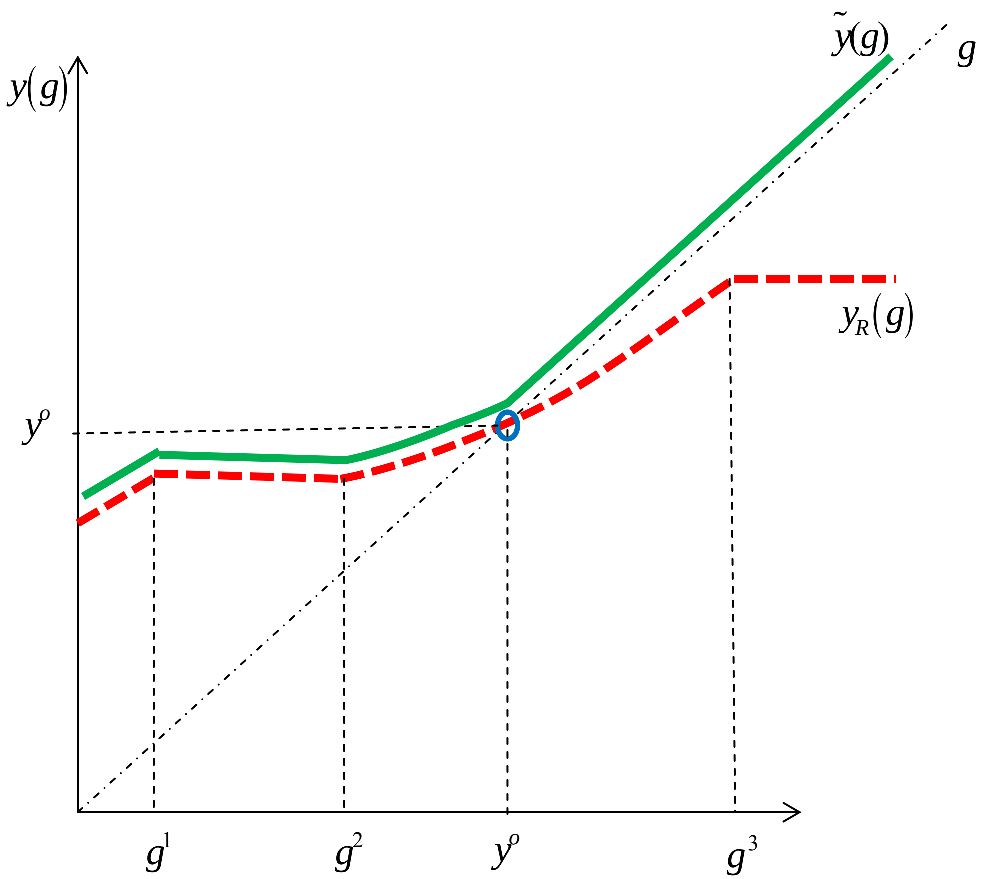
- In the best equilibrium, investment is increasing in n; in the worst equilibrium is decreasing in n.

- We have multiplicity even as $n \to \infty$:

- Multiplicity disappears in the "static version:" $\delta = 0$

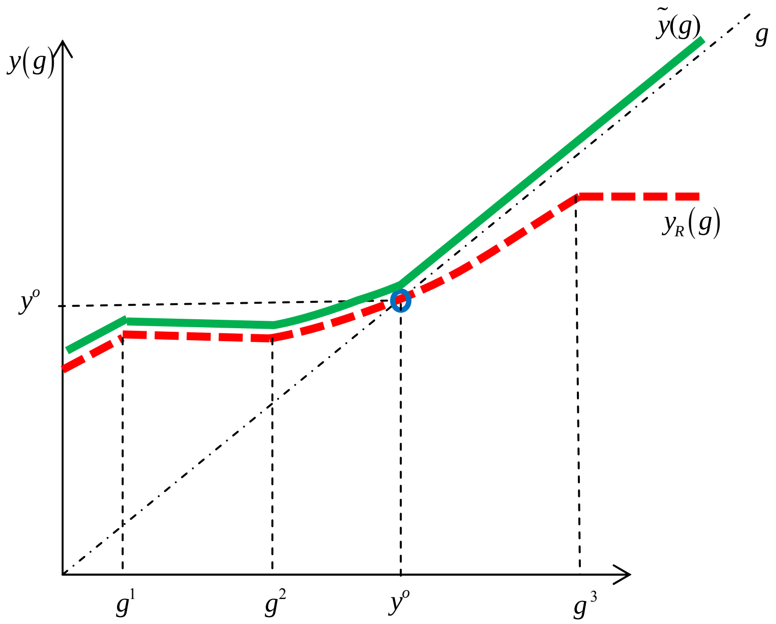
- As $\delta \to 1$, highest steady state $\to$ efficient level.

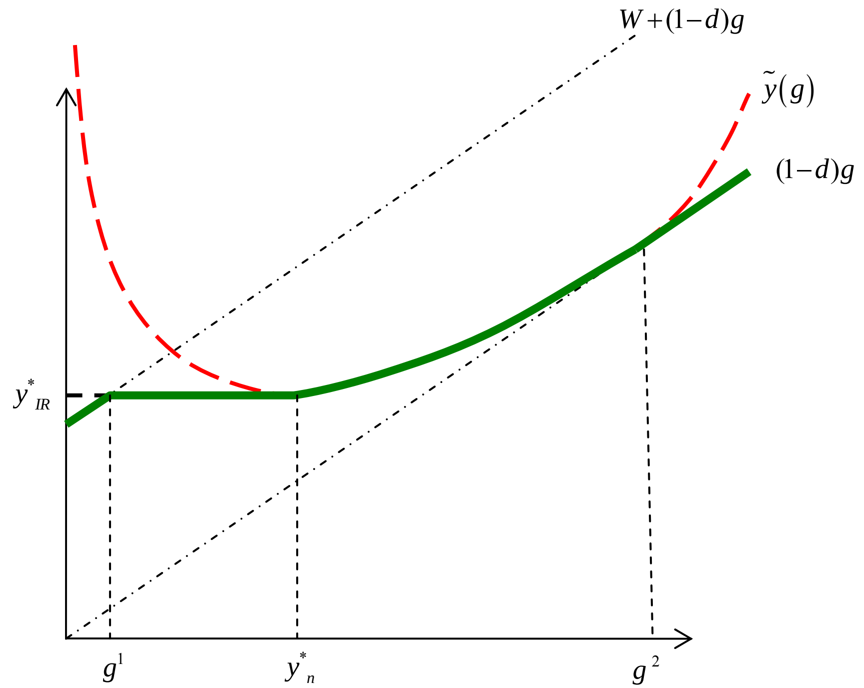
III. Equilibria in a IIE



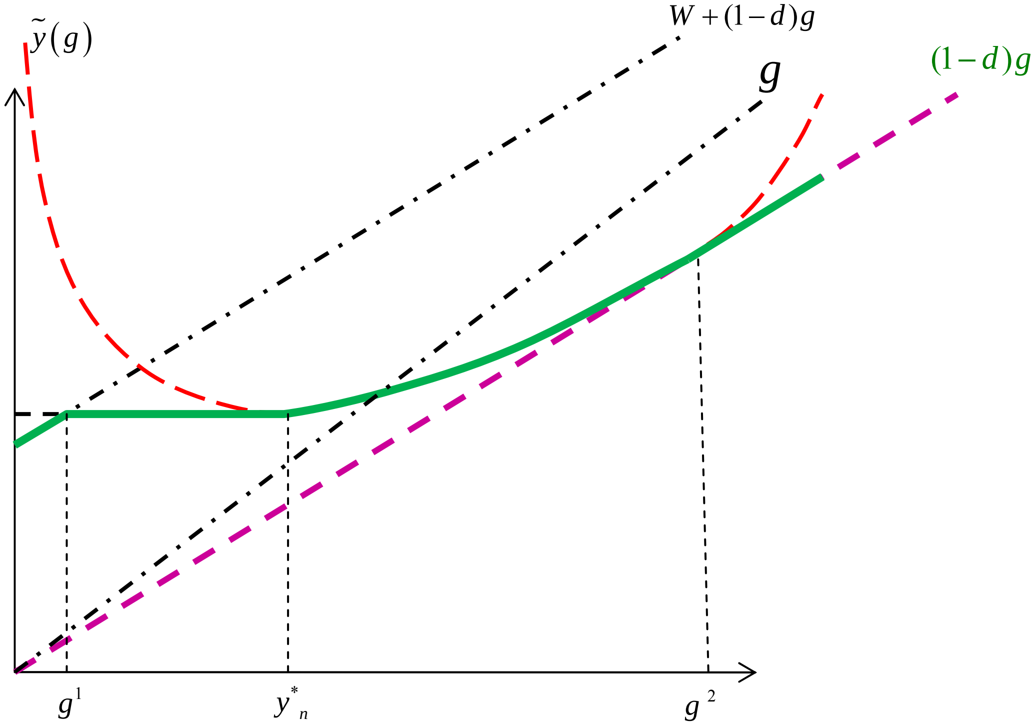
- Assume $d=0$.
- At y^o agent j does not have to fear his investment will be "stolen".
- Irreversibility is a commitment device.
- "Eq." is not concave at y^o.
- This has a ripple effect on states $g<y^o$.

- The equilibrium should merge *smoothly* with the constraint.





Proposition. *In a IIE there is a unique concave and monotonic eq. with investment function like this:*



- In general when *d* is high, we may have other non-concave equilibria (out of the equilibrium path).

- The extreme case is when *d=1*, in which case RIE and IIE are almost the same.

- **Proposition.** As *d→0, the set of equilibrium steady states in a IIE converges to the upperbound of the steady states of a RIE:*

$$y_{IR}^{o} = [u']^{-1}(1-\delta)$$

while the set of feasible steady states in a RIE is:

$$[u']^{-1}(1) \leq y_{R}^{o} \leq [u']^{-1}(1-\delta)$$

IV. Non-monotonic strategies and cycles

- Non monotonic equilibria always exist: the lowest steady state is lower; the highest is the same.

- Non-monotonic equilibria are interesting because:

  - Steady states with damped obscillations always exist;

  - We can have equilibria with no stable steady state, only persistent cycles.

V. Conclusion

- We have studied a model in which *n* infinitely lived agents choose between consumption and a durable public good.

- Two possible cases: reversible and irreversible economies:
  - In reversible economies there is a continuum of equilibria: in the best equilibrium the SS increases in *n*; in the worst equilibrium, it decreases in *n*.
  - In irreversible economies the set of SS converges to the best SS with reversibility as $d \rightarrow 0$.
  - There are non-monotonic equilibria in which g converges with damped oscillations; and in which there are limit cycles.